Th 1. Let $E \in \mathfrak{M}$, $E \subseteq (a,b) \subseteq \mathbb{R}$. Let $f : E \to \mathbb{R}$ (or $f = 0$ on $\mathbb{R} \setminus E$) be measurable; let $\varepsilon > 0$. Then $\exists$ simple function φ, step function ψ and continuous function g vanishing on $\mathbb{R} \setminus (a,b)$ such that

$$|f-\varphi|,\ |f-\psi|,\ |f-g| < \varepsilon$$

on $E \setminus A$ with some $A \in \mathfrak{M}$ of mea $< \varepsilon$.

Proof. I. Special Case when $m < f \leq M$ on E with some $m, M \in \mathbb{R}$. Let $n \in N$ be such that $\frac{M-m}{n} < \varepsilon$; divide the range (m, M) into n subintervals $J_i := (y_{i-1}, y_i]$, where $y_i = m + \frac{M-m}{n} i$, $y_0 = m$ $(i = 1, 2, \dots n)$, and let $f^{-1}(J_i) := \{x \in E : y_{i-1} < f(x) \leq y_i\}$ (measurable)

Define $\varphi := \sum_{i=1}^{n} y_{i-1} \chi_{f^{-1}(J_i)}$ (so $\varphi = 0$ on $\mathbb{R} \setminus E$; in particular $\varphi = 0$ at a, b)

Clearly $m < \varphi(x) < f(x) \leq M \; \forall x \in E$ and

$0 < f(x) - \varphi(x) \leq \frac{M-m}{n}$ (length of the subinterval) $< \varepsilon$.

Thus the result (regarding simple functions) holds (with empty exceptional set) in the special case. Now we apply the Corollary of Littlewood's 1st Principle to get step-functions ψ, and continuous functions g to approximate φ: they vanish outside (a,b) and are such that

$$|\varphi - \psi| < \varepsilon \text{ on } E \setminus A_1 \; (\text{so } |f-\psi| < 2\varepsilon \text{ on } E \setminus A_1)$$
$$|\varphi - g| < \varepsilon \text{ on } E \setminus A_2 \; (\text{so } |f-g| < 2\varepsilon \text{ on } E \setminus A_2)$$

with some "exceptional sets" A_1, A_2 of measures $< \varepsilon$.

For the general case, let

$$F_n := \{x \in E : |f(x)| \geq n\} \quad \left(\text{so } m \ni F_n \downarrow_n \; \bigcap_{n \in \mathbb{N}} F_n = \emptyset \text{ as } f(x) \in \mathbb{R} \; \forall x\right).$$

By the Monotone Convergence Lemma for measures, $\lim_n m(F_n) = 0$ and so $\exists N \in \mathbb{N}$ s.t. $m(F_N) < \varepsilon$.

Let $f_N : E \to [-N, N]$ ($\&$ vanishing outside E) be such that
$$f_N(x) = \begin{cases} f(x) & \text{if } |f(x)| < N, \\ 0 & \text{if } |f(x)| \geq N, \end{cases}$$

Then f_N satisfies the assumptions for the stated special case (in particular $|f_N| \leq N$) so $\exists$ the corresponding φ_N, ψ_N, g_N with exceptional sets $\emptyset$, A_1, A_2 (of mea $< \varepsilon$):

$$|f_N - \varphi_N| < \varepsilon \text{ on } E$$

$$|f_N - \psi_N| < 2\varepsilon \text{ on } E \setminus A_1^{(N)}$$

$$|f_N - g_N| < 2\varepsilon \text{ on } E \setminus A_2^{(N)}$$

(all vanish outside (a,b)). Since $f = f_N$ on $\mathbb{R} \setminus F_N$ and $m(F_N) < \varepsilon$, one has

$$|f - \varphi_N| < \varepsilon \text{ on } E \setminus A_3$$

$$|f - \psi_N| < 2\varepsilon \text{ on } E \setminus A_1$$

$$|f - g_N| < 2\varepsilon \text{ on } E \setminus A_2$$

where A_1, A_2, A_3 of measures $< 2\varepsilon$.

Note (Extension). $\psi: \mathbb{R} \to \mathbb{R}$ is called a step-function if it has at most countably many steps, and on any interval of finite length if it has at most finitely many steps. Similarly one modify the notion of simple functions.

Th 2. Let $f: E \to \mathbb{R}$ be measurable and $m(E) \lneq +\infty$. Then $\forall \varepsilon > 0$ $\exists$ $\begin{cases} \text{simple } \varphi \\ \text{step } \psi \\ \text{continuous } g \end{cases}$ from $\mathbb{R} \to \mathbb{R}$ and $m(A_i) < \varepsilon$ $(i = 1, 2, 3)$ s.t.

$$|f - \varphi| < \varepsilon \text{ on } E \setminus A_1$$

$$|f - \psi| < \varepsilon \text{ on } E \setminus A_2$$

$$|f - g| < \varepsilon \text{ on } E \setminus A_3,$$

provided that "simple" and "step-function" are of "extended meaning"

Proof. $\forall n \in \mathbb{Z}$, let $E_n := E \cap (n-1, n) \in \mathcal{M}$ and $E \subseteq \bigcup_{n \in \mathbb{Z}} E_n \cup \mathbb{Z}$. Note that $m(\mathbb{Z}) = 0$ and $\sum_{n \in \mathbb{Z}} \frac{1}{2^{|n|}} = 3 < 4$. By Th 1 (applied to $E_n \subset (n-1, n)$), $\exists$ (simple, step, cts) φ_n, ψ_n, g_n with $m(A_i^{(n)}) < \frac{\varepsilon}{4 \cdot 2^{|n|}}$ $(i = 1, 2, 3)$ such that all vanish outside $(n, n+1)$ and

$$|f - \varphi_n| < \varepsilon \text{ on } E_n \setminus A_1^{(n)}$$

$$|f - \psi_n| < \varepsilon \text{ on } E_n \setminus A_2^{(n)},$$

$$|f - g_n| < \varepsilon \text{ on } E_n \setminus A_3^{(n)}$$

Then $A_i := \bigcup_{n \in \mathbb{Z}} A_i^{(n)}$ is of mea $< \varepsilon$ $(i = 1, 2, 3)$

Define $\varphi, \psi, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(x) = \sum_{n \in \mathbb{Z}} \varphi_n(x) \quad \left(\text{actually all but one term are zero}\right), \ \forall x \in \mathbb{R}$$

$$\psi(x) := \cdots$$

$$g(\cdot) := \cdots$$

Note that g is cts but φ is not simple in the strict sense (φ is simple when restricted to any finite interval). Clearly

$$|f - \varphi| < \varepsilon \text{ on } E \setminus A_1$$

$$|f - \psi| < \varepsilon \text{ on } E \setminus A_2$$

$$|f - g| < \varepsilon \text{ on } E \setminus A_3$$